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NC Wilson lines and the inverse Seiberg–Witten map for non-degenerate star products

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Abstract. Open Wilson lines are known to be the observables of non-commutative gauge theory with Moyal–Weyl $\star$-product. We generalize these objects to more general $\star$-products. As an application we derive a formula for the inverse Seiberg–Witten map for $\star$-products with invertible Poisson structures

1 Introduction

Non-commutative gauge theories have been under close investigation as it was realized that they can be implemented by a certain string theory [8]. In these theories the non-commutativity is introduced via a $\star$-product on ordinary function spaces. Most research up to now has only considered the case of the Moyal–Weyl $\star$-product

$$f \star g = \lim_{x' \to x} \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{ij}\partial_i \partial'_j} f(x)(x')$$

which depends on a constant tensor θ^{ij}. In the string theory approach, this tensor is related to a constant B-field on a brane. On a curved brane this B-field becomes position dependent [25]. For this it is necessary to look at the case where the tensor θ^{ij} is not constant anymore. In this case $\star$-products can be defined as polydifferential operators:

$$f \star g = fg + \frac{\mathrm{i}}{2}\theta^{ij}(x)\partial_i f \partial_j g + \text{higher order terms.} \quad (1)$$

The functions should still form an associative algebra. Therefore θ^{ij} has to be a Poisson tensor:

$$\theta^{il}\partial_l \theta^{jk} + \text{cyc.} = 0.$$

On the other hand one can show that every Poisson tensor gives rise to a $\star$-product that looks like (1) [1] and that a large class of algebras may be represented in this way [21]. In these cases the higher order terms all depend only on θ^{ij} and its derivatives.

The fundamental objects of non-commutative gauge theory are covariant coordinates which can also be defined for the $\star$-products (1). In this paper we will use these covariant coordinates to generalize the open Wilson lines introduced in [24, 26, 27]. In [23] they were used to give an exact formula for the inverse Seiberg–Witten map. We will generalize this construction for $\star$-products of type (1) with invertible Poisson structure θ^{ij}.

2 Covariant coordinates

In a non-commutative version of a $U(1)$-gauge theory, a scalar field should transform like

$$\phi' = g \star \phi,$$

where g is a function that is invertible with respect to the $\star$-product:

$$g \star g^{-1} = g^{-1} \star g = 1.$$

Note that multiplication with a coordinate function is not covariant anymore:

$$\left(x^i \star \phi\right)' \neq x^i \star \phi'.$$

In the classical case the same problem arises with the partial derivatives. In analogy to this, covariant coordinates

$$X^i(x) = x^i + A^i(x)$$

can be introduced transforming in the adjoint representation

$$X^{i\prime} = g \star X^i \star g^{-1}.$$

Now the product of a covariant coordinate with a field is again a field. An infinitesimal version of this is presented in [4]. The equivalence of both approaches is investigated in [2].

In [8] it was shown that commutative and non-commutative gauge theory can be related by the so-called Seiberg–Witten map. Mapping the classical gauge transformations and gauge fields to their non-commutative counterparts, one can show that

$$A^i = \theta^{ij} a_j + \mathcal{O}\left(\theta^2\right). \quad (2)$$

This equality also holds in the case of the $\star$-products (1) [12].

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3 Wilson lines

In the case $\theta^{ij} = \text{const.}$ the basic observation was that translations in space are gauge transformations [24]. They are realized by

$$T_l x^j = x^j + l_i\theta^{ij} = \mathrm{e}_\star^{\mathrm{i}l_i x^i} \star f \star \mathrm{e}_\star^{-\mathrm{i}l_i x^i},$$

where $\mathrm{e}_\star$ is the $\star$-exponential. Every multiplication in its Taylor series is replaced by the $\star$-product. Note that in the constant case $\mathrm{e}_\star^{x^i} = \mathrm{e}^{x^i}$. Now one can pose the question what happens if one uses covariant coordinates. In this case the inner automorphism

$$f \to \mathrm{e}_\star^{\mathrm{i}l_i X^i} \star f \star \mathrm{e}_\star^{-\mathrm{i}l_i X^i}$$

should consist of a translation and a gauge transformation dependent on the translation. If we subtract the translation again only the gauge transformation remains and the resulting object

$$W_l = \mathrm{e}_\star^{\mathrm{i}l_i X^i} \star \mathrm{e}_\star^{-\mathrm{i}l_i x^i}$$

has a very interesting transformation behavior under a gauge transformation:

$$W_l'(x) = g(x) \star W_l(x) \star g^{-1}\left(x + l_i\theta^{ij}\right).$$

It transforms like a Wilson line starting at x and ending at $x + l\theta$.

As in the constant case we can start with

$$W_l = \mathrm{e}_\star^{\mathrm{i}l_i X^i} \star \mathrm{e}_\star^{-\mathrm{i}l_i x^i},$$

where $\star$ is now an arbitrary $\star$-product (1). The transformation property of W_l is now

$$W_l'(x) = g(x) \star W_l(x) \star g^{-1}(T_l x),$$

where

$$T_l x^j = \mathrm{e}_\star^{\mathrm{i}l_i x^i} \star x^j \star \mathrm{e}_\star^{-\mathrm{i}l_i x^i}$$

is an inner automorphism of the algebra, which can be interpreted as a quantized coordinate transformation. Note that the $\mathrm{e}_\star^{l_i X^i}$ do not close to form a group for $\theta^{ij}(x)$ at least quadratic in the x's. Therefore it is not clear how to generalize NC Wilson lines for arbitrary curves as in [24]. If we replace commutators by Poisson brackets the classical limit of these coordinate transformations may be calculated:

$$T_l x^k = \mathrm{e}_\star^{\mathrm{i}l_i [x^i \overset{\star}{,} \cdot]} x^k \approx \mathrm{e}^{-l_i\{x^i,\cdot\}} x^k = \mathrm{e}^{-l_i\theta^{ij}\partial_j} x^k,$$

the formula becoming exact for θ^{ij} constant or linear in x. We see that the classical coordinate transformation is the flow induced by the Hamiltonian vector field $-l_i\theta^{ij}\partial_j$. At the end we may expand W_l in terms of θ and get

$$W_l = \mathrm{e}^{\mathrm{i}l_i\theta^{ij}a_j} + \mathcal{O}\left(\theta^2\right),$$

where we have replaced A^i by its Seiberg–Witten expansion. We see that for l small this really is a Wilson line starting at x and ending at $x + l\theta$. For a given $\star$-product, the higher order corrections to this expression can in principle be calculated. Note that this expression would also depend on the specific choice of the Seiberg–Witten map of the covariant coordinates (2).

4 Observables

As space translations are included in the non-commutative gauge transformations no local observables can be constructed. In the case $\theta^{ij} = \text{const.}$ one has to integrate over the whole space

$$U_l = \int \mathrm{d}^{2n}x\, W_l(x) \star \mathrm{e}_\star^{\mathrm{i}l_i x^i}.$$

We will assume that θ^{ij} is invertible and therefore the dimension of the space has to be even: $N = 2n$. If one goes to the Fock space representation of the algebra one sees that this corresponds to

$$U_l = \operatorname{tr} \mathrm{e}^{\mathrm{i}l_i \hat{X}^i}.$$

In the more general case of non-constant θ^{ij} we therefore need a trace for the $\star$-product, i.e. a functional tr with the property

$$\operatorname{tr} f \star g = \operatorname{tr} g \star f.$$

Only in this case the trace of a covariant quantity becomes invariant. For many $\star$-products we know that the trace may be written as

$$\operatorname{tr} f = \int \mathrm{d}^{2n}x\, \Omega(x)\, f(x) \tag{3}$$

with a measure function Ω. Due to the cyclicity of the trace it has to fulfill

$$\partial_i\left(\Omega\theta^{ij}\right) = 0 \tag{4}$$

which can easily be calculated with (1). As we take the Poisson structure θ^{ij} to be invertible, the inverse of the Pfaffian

$$\frac{1}{\Omega} = \operatorname{Pf}(\theta) = \sqrt{\det(\theta)} = \frac{1}{2^n n!}\epsilon_{i_1 i_2 \ldots i_{2n}}\theta^{i_1 i_2}\ldots\theta^{i_{2n-1}i_{2n}}$$

is a solution to this equation. If (3) is fulfilled, cyclicity is only guaranteed to first order. In principle we have to calculate higher orders of Ω according to the $\star$-product chosen. Nevertheless there can always be found a $\star$-product so that a measure function fulfilling (4) guarantees cyclicity to all orders [7]. Now we are able to write down a large class of observables for the above defined non-commutative gauge theory, namely

$$U_l = \int \mathrm{d}^{2n}x\, \Omega(x)\, W_l(x) \star \mathrm{e}_\star^{\mathrm{i}l_i x^i} = \int \mathrm{d}^{2n}x\, \Omega(x)\, \mathrm{e}_\star^{\mathrm{i}l_i X^i(x)},$$

or more general

$$f_l = \int \mathrm{d}^{2n}x\, \Omega(x)\, f(X^i) \star \mathrm{e}_\star^{\mathrm{i}l_i X^i(x)},$$

with f an arbitrary function of the covariant coordinates.

5 Inverse Seiberg–Witten map

As an application of the above constructed observables we generalize [23] to arbitrary $\star$-products, i.e. we give a formula for the inverse Seiberg–Witten map for $\star$-products with invertible Poisson structure. In order to map non-commutative gauge theory to its commutative counterpart we need a functional $f_{ij}[X]$ fulfilling

$$f_{ij}\left[g\star X\star g^{-1}\right]=f_{ij}[X],$$

$$\mathrm{d}f=0$$

and

$$f_{ij}=\partial_i a_j-\partial_j a_i+\mathcal{O}(\theta).$$

f is a classical field strength and reduces in the limit $\theta\to 0$ to the correct expression.

To prove the first and the second property we will only use the algebra properties of the $\star$-product and the cyclicity of the trace. All quantities with a hat will be elements of an algebra. With this convention let $\hat{X}^i$ be covariant coordinates in an algebra, transforming under gauge transformations like

$$\hat{X}^{i\prime}=\hat{g}\hat{X}^i\hat{g}^{-1},$$

with $\hat{g}$ an invertible element of the algebra. Now define

$$\hat{F}^{ij}=-\mathrm{i}\left[\hat{X}^i,\hat{X}^j\right]$$

and

$$\left(\hat{F}^{n-1}\right)_{ij}=\frac{1}{2^{n-1}(n-1)!}\epsilon_{iji_1i_2\dots i_{2n-2}}\hat{F}^{i_1i_2}\dots\hat{F}^{i_{2n-3}i_{2n-2}}.$$

Note that the space is $2n$ dimensional. The expression

$$\mathcal{F}_{ij}(k)=\mathrm{str}_{\hat{F},\hat{X}}\left(\left(\hat{F}^{n-1}\right)_{ij}\mathrm{e}^{\mathrm{i}k_j\hat{X}^j}\right)\tag{5}$$

clearly fulfills the first property due to the properties of the trace. str is the symmetrized trace, i.e.

$$\mathrm{str}_{\hat{F},\hat{X}}\left(\hat{F}^q\hat{X}^r\right)=\frac{q!r!}{(q+r)!}\mathrm{tr}\Big(\hat{F}^q\hat{X}^r$$

$$+\ \text{all other possible permutations of } q\ \hat{F}'\text{s and } r\ \hat{X}'s\Big);$$

see also [23]. Note that symmetrization is only necessary for space dimension bigger than 4 due to the cyclicity of the trace. In dimensions 2 and 4 we may replace str by the ordinary trace tr. $\mathcal{F}_{ij}(k)$ is the Fourier transform of a closed form if

$$k_{[i}\mathcal{F}_{jk]}=0,$$

or if the current

$$J^{i_1\dots i_{2n-2}}=\mathrm{str}_{\hat{F},X}\left(\hat{F}^{[i_1i_2}\dots\hat{F}^{i_{2n-3}i_{2n-2}]}\mathrm{e}^{\mathrm{i}k_j\hat{F}^j}\right)$$

is conserved, respectively:

$$k_iJ^{i\dots}=0.$$

This is easy to show, if one uses

$$\mathrm{str}_{\hat{F},\hat{X}}\left(\left[k_i\hat{X}^i,\hat{X}^l\right]\mathrm{e}^{\mathrm{i}k_j\hat{X}^j}\dots\right)$$
$$=\mathrm{str}_{\hat{F},\hat{X}}\left(\left[\hat{X}^l,\mathrm{e}^{\mathrm{i}k_j\hat{X}^j}\right]\dots\right)=\mathrm{str}_{\hat{F},\hat{X}}\left(\mathrm{e}^{\mathrm{i}k_j\hat{X}^j}\left[\hat{X}^l,\dots\right]\right),$$

which can be calculated by simple algebra.

To prove the last property we have to switch to the $\star$-product formalism and expand the formula in θ^{ij}. The expression (5) now becomes

$$\mathcal{F}[X]_{ij}(k)=\int\frac{\mathrm{d}^{2n}x}{\mathrm{Pf}(\theta)}\left((F_\star^{n-1})_{ij}\star\mathrm{e}_\star^{\mathrm{i}k_jX^j}\right)_{\mathrm{sym}\,F,X}.$$

The expression in brackets has to be symmetrized in F^{ij} and X^i for $n>2$. Up to third order in θ^{ij}, the commutator F^{ij} of two covariant coordinates is

$$\begin{aligned}F^{ij}&=-\mathrm{i}\left[X^i\stackrel{\star}{,}X^j\right]\\&=\theta^{ij}-\theta^{ik}f_{kl}\theta^{lj}-\theta^{kl}\partial_l\theta^{ij}a_k+\mathcal{O}(3),\end{aligned}$$

with $f_{ij}=\partial_ia_j-\partial_ja_i$ the ordinary field strength. Furthermore we have

$$\mathrm{e}_\star^{\mathrm{i}k_iX^i}=\mathrm{e}^{\mathrm{i}k_ix^i}\left(1+\mathrm{i}k_i\theta^{ij}a_j\right)+\mathcal{O}(2).$$

If we choose the antisymmetric $\star$-product (1), the symmetrization will annihilate all the first order terms of the $\star$-products between the F^{ij} and X^i, and therefore we get

$$\begin{aligned}&-\mathcal{F}[X]_{ij}(k)\\&=-2n\int\frac{\mathrm{d}^{2n}x}{\epsilon\theta^n}\Big(\epsilon_{ij}\theta^{n-1}-(n-1)\epsilon_{ij}\theta^{n-2}\theta f\theta\\&\qquad-\theta^{kl}\partial_l(\epsilon_{ij}\theta^{n-1})a_k\Big)\mathrm{e}^{\mathrm{i}k_ix^i}+\mathcal{O}(1)\\&=-2n\int\frac{\mathrm{d}^{2n}x}{\epsilon\theta^n}\Big(\epsilon_{ij}\theta^{n-1}-(n-1)\epsilon_{ij}\theta^{n-2}\theta f\theta\\&\qquad-\frac{1}{2}\epsilon_{ij}\theta^{n-1}f_{kl}\theta^{kl}\Big)\mathrm{e}^{\mathrm{i}k_ix^i}+\mathcal{O}(1)\\&=\int\mathrm{d}^{2n}x\,\Big(\theta^{-1}_{ij}+2n(n-1)\frac{\epsilon_{ij}\theta^{n-2}\theta f\theta}{\epsilon\theta^n}\\&\qquad-\frac{1}{2}\theta^{-1}_{ij}f_{kl}\theta^{kl}\Big)\,\mathrm{e}^{\mathrm{i}k_ix^i}+\mathcal{O}(1),\end{aligned}$$

using partial integration and $\partial_i\left(\epsilon\theta^n\theta^{ij}\right)=0$. To simplify the notation we introduced

$$\epsilon_{ij}\theta^{n-1}=\epsilon_{iji_1j_1\dots i_{n-1}j_{n-1}}\theta^{i_1j_1}\dots\theta^{i_{n-1}j_{n-1}}\ \text{etc.}$$

In the last line we have used

$$\theta^{-1}_{ij}=-\frac{\left(\theta^{n-1}\right)_{ij}}{\mathrm{Pf}(\theta)}=-2n\frac{\epsilon_{ij}\theta^{n-1}}{\epsilon\theta^n}.$$

We will now have a closer look at the second term, noting that

$$\theta^{ij}\frac{\epsilon_{ij}\theta^{n-2}\theta f\theta}{\epsilon\theta^n} = -\frac{1}{2n}\theta^{-1}_{kl}\theta^{kr}f_{rs}\theta^{sl} = -\frac{1}{2n}f_{rs}\theta^{rs},$$

and therefore

$$\frac{\epsilon_{ij}\theta^{n-2}\theta f\theta}{\epsilon\theta^n} = a\frac{\epsilon_{ij}\theta^{n-1}}{\epsilon\theta^n}f_{rs}\theta^{rs} + bf_{ij}, \qquad (6)$$

with $a+b=-\frac{1}{2n}$. Taking e.g. $i=1, j=2$ we see that

$$\epsilon_{12\ldots kl}\theta^{n-2}\theta^{kr}f_{rs}\theta^{sl} = \epsilon_{12\ldots kl}\theta^{n-2}\left(\theta^{k1}\theta^{2l}-\theta^{k2}\theta^{1l}\right)f_{12} + \text{terms without } f_{12}.$$

Especially there are no terms involving $f_{12}\theta^{12}$ and we get for the two terms on the right hand side of (6)

$$2a\epsilon_{12}\theta^{n-1}f_{12}\theta^{12} = -2nb\epsilon_{12}\theta^{12}\theta^{n-1}f_{12},$$

and therefore $b=-\frac{a}{n}$. This has the solution

$$a=-\frac{1}{2(n-1)} \quad \text{and} \quad b=\frac{1}{2n(n-1)}.$$

With the resulting

$$2n(n-1)\frac{\epsilon_{ij}\theta^{n-2}\theta f\theta}{\epsilon\theta^n} = \frac{1}{2}\theta^{-1}_{ij}f_{kl}\theta^{kl} + f_{ij}$$

we finally get

$$-\mathcal{F}[X]_{ij}(k) = \int \mathrm{d}^{2n}x\,\left(\theta^{-1}_{ij}+f_{ij}\right)\mathrm{e}^{\mathrm{i}k_i x^i} + \mathcal{O}(1).$$

Therefore

$$f[X]_{ij} = \mathcal{F}[X]_{ij}(k) - \mathcal{F}[x]_{ij}(k)$$

is a closed form that reduces in the classical limit to the classical Abelian field strength. We have found an expression for the inverse Seiberg–Witten map.

6 Outlook

It would be interesting to find an expression similar to (5) for other non-commutative (compact) spaces like the fuzzy torus and the fuzzy sphere. In the second case we would be able to map a commutative $su(2)$-gauge theory to an commutative Abelian gauge theory in higher dimensions.

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